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THE INSCRIBED AND CIRCUMSCRIBED SQUARES OF A QUADRILATERAL AND THEIR SIGNIFICANCE IN KINEMATIC GEOMETRY.

By C. M. HEBBERT.

Professor Emch has shown that in any closed convex curve at least one square may be inscribed* and that about such a curve at least one square may be circumscribed. Moreover, the results may be extended to convex polygons formed by analytic arcs.

This investigation leads to methods for actually constructing the inscribed squares of all regular polygons and any quadrilateral and the circumscribed square of any quadrangle. By inscribed square is meant one whose vertices lie on four sides of the figure and by circumscribed square is meant a square each of whose vertices passes through a vertex of the quadrangle. Because of the number of interesting propositions of kinematic geometry resulting from the solution of these problems for the quadrilateral and quadrangle, I shall confine myself to their discussion.

THEOREM I. *In every quadrilateral may be inscribed at least one square having a vertex on each of the four sides. If there is more than one such square in a given quadrilateral, there is an infinite number.*

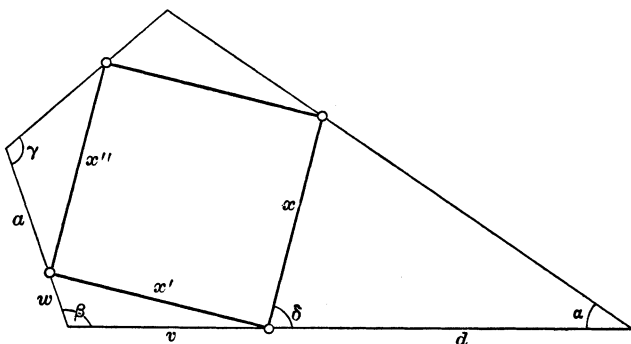


FIG. 1.

For let x be the side of a square having its vertices on the four sides of a given quadrilateral and let δ be the angle made by the side x and the side d of the quadrilateral. Also let α , β , γ be the angles of the quadrilateral as indicated and let a be the side included by the angles β and γ . v and w are the segments cut off from d and a , respectively, by corresponding vertices of the square.

* "Some Properties of Closed Convex Curves in a Plane," American Journal of Mathematics, vol. 35, pp. 407-412.

Then making use of the law of sines and the conditions that $x = x' = x''$ and $x \perp x'$ and $x' \perp x''$, we get

$$\tan \delta = \frac{d(\csc \beta + \cos \beta + \cot \gamma \sin \beta) - a(1 + \cot \beta)}{d(\cot \gamma \cos \beta - \sin \beta) + a(1 + \cot \alpha)}.$$

This equation shows that in general there is always one real value for δ in a given quadrilateral. The equation can have more than one solution if and only if the right-hand member is $0/0$ and then there are an infinite number of solutions, according to a theorem stated by A. Hurwitz:*

"If it is possible in special cases to get more than n solutions for a problem which in general has n solutions determined by the roots of an equation of the n th degree, then in these special cases there are an infinite number of solutions."

Since V and W are linear in δ , every point of the side of a quadrilateral in which an infinite number of squares can be inscribed is the vertex of one and only one inscribed square.

THEOREM II. *If three vertices of a variable square move along three fixed straight lines, the fourth vertex moves along a fixed straight line and the centers are collinear.*

For if each of two squares has three of its vertices on three fixed lines, the line joining the fourth vertices will form with the three fixed lines a quadrilateral in which two squares are inscribed. Hence in the quadrilateral so formed an infinite number of squares may be inscribed and the fourth side so determined is the line along which the fourth vertex moves.

Differentiation and division of the linear expressions for V and W in Fig. 1, gives $dV/dW = \text{a constant}$, i. e. the ratio of the distances through which these two vertices move is constant. Similarly for any pair of vertices.

Let ABC and $A'B'C'$ (Fig. 2) be three positions of opposite vertices. Then AA' , BB' , CC' will be diagonals. For the sake of simplicity, assume that $AB = BC$ and $A'B' = B'C'$. Let F , G , H be the mid-points of AA' , BB' , CC' respectively.

Then $FBHB'$ is a parallelogram and since the diagonal FH bisects the diagonal BB' , F , G , and H are collinear. Also $FG = GH$.

Kinematically, these results may be expressed as follows:

THEOREM III. *If two vertices of a variable square move with constant velocities along two fixed straight lines, the other two vertices move with constant velocities along two fixed lines and the center moves with constant velocity along a straight line.*

Some special cases of this theorem have been stated by E. Study, who proved them by means of the "real pictures" of imaginary points.†

* Math. Annalen, vol. XV, p. 8.

† E. Study, Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: Ebene analytische Kurven und zu ihnen gehörige Abbildungen, p. 16.

to EF and EG , respectively, are equal and determine a square having three vertices on three sides of the given quadrilateral.

Repetition of this process gives a second square having three vertices on the same three sides of the given quadrilateral.

Now if we think of this second square as being obtained by allowing the vertices of the first square to move along the three straight lines DA , AB , and BC , while the sides vary in length, we see that by Theorem II, the fourth vertex must have moved along the fixed straight line PR . Then the sect-point, Q , of PR and DC will be a vertex of the square having its four vertices on the four sides of the quadrilateral. The other three vertices can be easily determined since by Theorem III, $PQ/PR = E'X/E'E''$, etc.

Since the vertices of the variable square in Theorems II and III move with constant velocities along the four sides of the quadrilateral, the sides

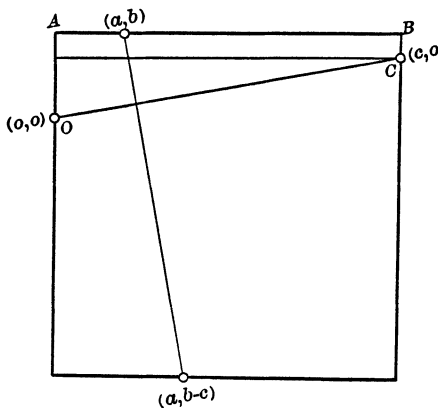


FIG. 4.

of the variable square envelope four parabolas. Consideration of the similar triangles formed by joining the focus of one of the parabolas to the vertices of two of the squares gives

THEOREM IIIa. *The four sides of the variable square whose four vertices move along four fixed straight lines envelope four confocal parabolas whose directrices form a square.*

THEOREM IV. *If three sides of a variable square pass through three fixed points, the fourth side passes through a fixed point.*

Given three points $(0, 0)$, (a, b) and $(c, 0)$ and through these points the lines $y = nx$, $y = n(x - c)$, and $n(y - b) + x - a = 0$.

Since AB is $\perp$ to AO and BC , $AB = OC \sin AOB = cn/\sqrt{1 + n^2}$.

Now in order for our fourth line to form a square with the three given lines, it must be at the distance $\pm cn/\sqrt{1 + n^2}$ from AB and be parallel to AB .

The distance of any point (x_1, y_1) from the line $n(y - b) + (x - a) = 0$ is

$$\frac{ny_1 + x_1 - nb - a}{\sqrt{1 + n^2}}.$$

Since we want (x_1, y_1) to be at distance $\pm cn / \sqrt{1 + n^2}$ from $n(y - b) + (x - a) = 0$, we have as equation of locus of (x_1, y_1)

$$\frac{ny_1 + x_1 - nb - a}{\sqrt{1 + n^2}} = \frac{\pm cn}{\sqrt{1 + n^2}}$$

or,

$$n(y_1 - b \pm c) + (x_1 - a) = 0$$

which may be written

$$\frac{y_1 - (b \pm c)}{x_1 - a} = -\frac{1}{n}.$$

This last form of the equation shows that the fourth side of the square through the three given points always passes through the point $(a, b \pm c)$.

It will be observed that the points (a, b) and $(a, b \pm c)$ are on the line $x = a$ which is $\perp$ to the x axis, and that the distance between these two points is equal to c .

Hence we have

COROLLARY 1. *A necessary and sufficient condition that it be possible to circumscribe an infinite number of squares about a given quadrangle is that the diagonals be perpendicular and equal.*

Therefore, to construct the circumscribed square of the quadrangle $ABCD$, draw $BE \perp$ and equal to AC . Through B draw a parallel to ED and through A and C draw perpendiculars to ED . These three lines and ED form the required square.*

The kinematic expression of theorem IV is:

THEOREM V. *If one side of a variable square turns about a fixed point while the extremities of this side move with constant velocities on two circles which pass through the fixed point, then the other three sides of the square turn about fixed points and the two remaining vertices of the square move with constant velocities on two circles which intersect each other and the two given circles in the remaining fixed points. The centers of the four circles form a square. The center of the variable square moves on a circle which passes through the sect-point and the mid-points of the diagonals joining the four fixed points. The velocities of the vertices and the center of the variable square are proportional to the radii of the corresponding circles, i. e. the angular velocities of vertices and center are constant.†*

*For another construction, see *Journal für die reine und angewandte Mathematik*, vol. XXXIV, p. 281.

†See Study, loc. cit., and A. Mannheim: *Principes et Développement de Géométrie Cinématique*, pp. 14-17.